

The annular decay property and capacity estimates for thin annuli

Anders Björn

*Department of Mathematics, Linköping University,
SE-581 83 Linköping, Sweden; anders.bjorn@liu.se*

Jana Björn

*Department of Mathematics, Linköping University,
SE-581 83 Linköping, Sweden; jana.bjorn@liu.se*

Juha Lehrbäck

*Department of Mathematics and Statistics, University of Jyväskylä,
P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland; juha.lehrback@jyu.fi*

Abstract. We obtain upper and lower bounds for the nonlinear variational capacity of thin annuli in weighted \mathbf{R}^n and in metric spaces, primarily under the assumptions of an annular decay property and a Poincaré inequality. In particular, if the measure has the 1-annular decay property at x_0 and the metric space supports a pointwise 1-Poincaré inequality at x_0 , then the upper and lower bounds are comparable and we get a two-sided estimate for thin annuli centred at x_0 , which generalizes the known estimate for the usual variational capacity in unweighted \mathbf{R}^n . Most of our estimates are sharp, which we show by supplying several key counterexamples. We also characterize the 1-annular decay property.

Key words and phrases: Annular decay property, capacity, doubling measure, metric space, Newtonian space, Poincaré inequality, Sobolev space, thin annulus, upper gradient, variational capacity, weighted \mathbf{R}^n .

Mathematics Subject Classification (2010): Primary: 31E05; Secondary: 30L99, 31C15, 31C45.

1. Introduction

We assume throughout the paper that $1 \leq p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. We also let $x_0 \in X$ be a fixed but arbitrary point and $B_r = B(x_0, r) = \{x : d(x, x_0) < r\}$.

In this paper, we continue the study of sharp estimates for the variational capacity $\text{cap}_p(B_r, B_R)$, which we started in Björn–Björn–Lehrbäck [5]. Therein we concentrated on the case $0 < 2r \leq R$, while in the present work we are interested in the case where the annulus $B_R \setminus B_r$ is thin, that is, $0 < \frac{1}{2}R \leq r < R$.

Assume for a moment that the measure μ is doubling and that the space X supports a p -Poincaré inequality. Then it is well known that $\text{cap}_p(B_r, B_{2r}) \simeq \mu(B_r)r^{-p}$ holds for all $0 < r < \frac{1}{8}\text{diam } X$. If in addition the exponents $0 < q \leq q' <$

∞ are such that

$$\left(\frac{r}{R}\right)^{q'} \lesssim \frac{\mu(B_r)}{\mu(B_R)} \lesssim \left(\frac{r}{R}\right)^q, \quad \text{if } 0 < r \leq R < \text{diam } X, \quad (1.1)$$

then, by [5, Theorem 1.1],

$$\text{cap}_p(B_r, B_R) \simeq \begin{cases} \mu(B_r)r^{-p}, & \text{if } p < q, \\ \mu(B_R)R^{-p}, & \text{if } p > q', \end{cases} \quad (1.2)$$

when $0 < 2r \leq R < \frac{1}{4} \text{diam } X$. However, when r is close to R these estimates are no longer valid; in particular, typically $\text{cap}_p(B_r, B_R) \rightarrow \infty$ when $r \rightarrow R$ and $p > 1$ (see Section 5 for more on when this holds). Moreover, the difference in the growth bounds in (1.1) does not play any role when r is close to R , and so it is obvious that other properties of the space determine the capacities of thin annuli.

In (unweighted) \mathbf{R}^n the following equalities hold for capacities of annuli for all $0 < r < R < \infty$ (see e.g. [13, p. 35]):

$$\text{cap}_p(B_r, B_R) = \begin{cases} C(n, p) |R^{(p-n)/(p-1)} - r^{(p-n)/(p-1)}|^{1-p}, & \text{if } p \notin \{1, n\}, \\ C(n, p) \left(\log \frac{R}{r}\right)^{1-n}, & \text{if } p = n, \\ C(n, p) r^{n-1}, & \text{if } p = 1. \end{cases}$$

When $0 < \frac{1}{2}R \leq r < R$, these yield the estimate

$$\text{cap}_p(B_r, B_R) \simeq \left(1 - \frac{r}{R}\right)^{1-p} \frac{m(B_R)}{R^p}, \quad (1.3)$$

where m is the n -dimensional Lebesgue measure.

The main goal in this paper is to find general conditions for the space X under which estimates similar to (1.3) hold. One such condition is the following measure decay property, which will play a crucial role in our results.

Definition 1.1. Let $\eta > 0$. The measure μ has the η -annular decay (η -AD) property at $x \in X$, if there is a constant C such that for all radii $0 < r < R$ we have

$$\mu(B(x, R) \setminus B(x, r)) \leq C \left(1 - \frac{r}{R}\right)^\eta \mu(B(x, R)). \quad (1.4)$$

If there is a common constant C such that (1.4) holds for all $x \in X$ (and all radii $0 < r < R$), then μ has the *global η -AD property*.

For most of the results in this paper it will be enough to require a pointwise AD property at x_0 , often together with pointwise versions of (reverse) doubling and Poincaré inequalities. This resembles the situation in [5], where for capacity estimates for nonthin annuli, such as (1.2), it was enough to require doubling (and reverse-doubling) and Poincaré inequalities to hold pointwise.

The global AD property was introduced (under the name volume regularity property) in Colding–Minicozzi [8, p. 125] for manifolds and independently by Buckley [7], who called it the annular decay property, for general metric spaces. A variant of the global AD property was already used in David–Journé–Semmes [9, p. 41]. Later, the global AD-property has been used by many other authors. See e.g. Buckley [7] and Routin [19] for more information and applications of the global AD property. We have not seen any considerations related to the pointwise AD property in the literature.

If X is a length space and μ is globally doubling, then μ has the global η -AD property for some $\eta > 0$, see the proof of Lemma 3.3 in [8]. Example 7.2 shows that the length space assumption cannot be dropped.

The AD property implies the following upper bound for the variational capacity.

Proposition 1.2. *Assume that μ has the η -AD property at x_0 . Then*

$$\text{cap}_p(B_r, B_R) \lesssim \left(1 - \frac{r}{R}\right)^{\eta-p} \frac{\mu(B_R)}{R^p}, \quad \text{if } 0 < r < R. \quad (1.5)$$

If μ has the global η -AD property, then the implicit constant is independent of x_0 .

The proof of this result is quite simple, and it is perhaps more interesting that there are similar lower bounds and that the estimate is sharp, as we show in Example 3.3. The sharpness is true even if one assumes that μ has the global η -AD property.

Lower bounds for capacities are in general considerably more difficult to obtain than upper bounds. Here we use relatively simple means to obtain lower bounds similar to the upper bounds, so that we obtain two-sided estimates as in (1.3). The key assumption is, as usual, some type of Poincaré inequality. When both the 1-AD property and the 1-Poincaré inequality are available, our upper and lower bounds coincide, and we obtain the following generalization of (1.3), which is our main result.

Theorem 1.3. *Assume that X supports a global 1-Poincaré inequality and that μ has the global 1-AD property. Then*

$$\text{cap}_p(B_r, B_R) \simeq \left(1 - \frac{r}{R}\right)^{1-p} \frac{\mu(B_R)}{R^p}, \quad \text{if } 0 < \frac{R}{2} \leq r < R \leq \frac{\text{diam } X}{3}. \quad (1.6)$$

As in Proposition 1.2 it is actually enough to require pointwise versions of the assumptions, but then the result is a bit more complicated to formulate; see Theorem 4.3 for the exact statement. Nevertheless, even with global assumptions the parameters are sharp, see Example 7.1. A different type of two-sided estimate for capacity is obtained in Theorem 4.4.

The 1-AD property, which Buckley [7] calls “strong annular decay”, is essential in both the upper and lower bounds of Theorem 1.3. The 1-AD property is even locally the best possible AD property, under very mild assumptions, see Proposition 3.4. To further illustrate this useful property, we establish several characterizations of the 1-AD property in Section 6.

Acknowledgement. A. B. and J. B. were supported by the Swedish Research Council. J. L. was supported by the Academy of Finland (grant no. 252108) and the Väisälä Foundation of the Finnish Academy of Science and Letters. Part of this research was done during several visits of J. L. to Linköping University in 2012–15, and one visit of A. B. to the University of Jyväskylä in 2015.

2. Preliminaries

In this section we introduce the necessary background notation on metric spaces and in particular on Sobolev spaces and capacities in metric spaces. See the monographs Björn–Björn [2] and Heinonen–Koskela–Shanmugalingam–Tyson [15] for more extensive treatments of these topics, including proofs of most of the results mentioned in this section.

A *curve* is a continuous mapping from an interval, and a *rectifiable* curve is a curve with finite length. We will only consider curves which are nonconstant, compact and rectifiable, and thus each curve can be parameterized by its arc length ds . The metric space X is a *length space* if whenever $x, y \in X$ and $\varepsilon > 0$, there is a curve between x and y with length less than $(1 + \varepsilon)d(x, y)$.

A property is said to hold for *p -almost every curve* if it fails only for a curve family Γ with zero p -modulus, i.e. there exists $0 \leq \rho \in L^p(X)$ such that $\int_\gamma \rho ds =$

∞ for every curve $\gamma \in \Gamma$. Following Heinonen–Koskela [14], we introduce upper gradients as follows (they called them very weak gradients).

Definition 2.1. A Borel function $g: X \rightarrow [0, \infty]$ is an *upper gradient* of a function $f: X \rightarrow [-\infty, \infty]$ if for all curves $\gamma: [0, l_\gamma] \rightarrow X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds, \quad (2.1)$$

where the left-hand side is considered to be ∞ whenever at least one of the terms therein is infinite. If $g: X \rightarrow [0, \infty]$ is measurable and (2.1) holds for p -almost every curve, then g is a *p -weak upper gradient* of f .

The p -weak upper gradients were introduced in Koskela–MacManus [18]. It was also shown there that if $g \in L^p(X)$ is a p -weak upper gradient of f , then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j \rightarrow g$ in $L^p(X)$. If f has an upper gradient in $L^p(X)$, then it has an a.e. unique *minimal p -weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p -weak upper gradient $g \in L^p(X)$ of f we have $g_f \leq g$ a.e., see Shanmugalingam [21] and Hajłasz [11]. Following Shanmugalingam [20], we define a version of Sobolev spaces on the metric measure space X .

Definition 2.2. For a measurable function $f: X \rightarrow [-\infty, \infty]$, let

$$\|f\|_{N^{1,p}(X)} = \left(\int_X |f|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients g of f . The *Newtonian space* on X is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The quotient space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [20]. In this paper we assume that functions in $N^{1,p}(X)$ are defined everywhere, not just up to an equivalence class in the corresponding function space. This is needed for the definition of upper gradients to make sense. If $f, h \in N_{\text{loc}}^{1,p}(X)$, then $g_f = g_h$ a.e. in $\{x \in X : f(x) = h(x)\}$, in particular $g_{\min\{f,c\}} = g_f \chi_{\{f < c\}}$ for $c \in \mathbf{R}$.

Definition 2.3. The *Sobolev p -capacity* of an arbitrary set $E \subset X$ is

$$C_p(E) = \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E .

The Sobolev capacity is countably subadditive and it is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if they differ only in a set of capacity zero. Moreover, if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u \sim v$. This is the main reason why, unlike in the classical Euclidean setting, we do not need to require the functions admissible in the definition of capacity to be 1 in a neighbourhood of E . In (weighted or unweighted) \mathbf{R}^n , C_p is the usual Sobolev capacity and $N^{1,p}(\mathbf{R}^n)$ and $N^{1,p}(\Omega)$ are the refined Sobolev spaces as in Heinonen–Kilpeläinen–Martio [13, p. 96], see Björn–Björn [2, Theorem 6.7 (ix) and Appendix A.2].

Definition 2.4. The measure μ is *doubling at x* if there is a constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad \text{whenever } r > 0. \quad (2.2)$$

If (2.2) holds with the same constant $C > 0$ for all $x \in X$, we say that μ is (*globally*) *doubling*.

We also say that the measure μ is *reverse-doubling at x* , if there are constants $\gamma, \tau > 1$ such that

$$\mu(B(x, \tau r)) \geq \gamma \mu(B(x, r)) \quad \text{for all } 0 < r \leq \text{diam } X/2\tau,$$

and that the measure μ is *Ahlfors Q -regular* if $\mu(B(x, r)) \simeq r^Q$ for all $x \in X$ and all $r > 0$.

The global doubling condition is often assumed in the metric space literature, but for many of our estimates it will be enough to assume that μ is doubling at x . If X is connected, or more generally uniformly perfect (see Heinonen [12]), and μ is globally doubling, then μ is also reverse-doubling at every point, with uniform constants. In the connected case, one can choose $\tau > 1$ arbitrarily and find $\gamma > 1$ independent of x , see e.g. Corollary 3.8 in [2]. If μ is merely doubling at x , then the reverse-doubling at x does not follow automatically and has to be imposed separately whenever needed.

The η -AD property at x_0 easily implies that μ is doubling at x_0 . The converse is not true even if X is a length space, as seen by considering $m + \delta_1$ on \mathbf{R} , where δ_1 is the Dirac measure at 1, which is doubling at 0, but does not have the η -AD property at 0 for any $\eta > 0$. (For an absolutely continuous example, consider \mathbf{R} equipped with the measure $w dx$, where $w(x) = \max\{1, 1/|x-1|(\log|x-1|)^2\}$.)

Definition 2.5. We say that X supports a *p -Poincaré inequality at x* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B = B(x, r)$, all integrable functions f on X , and all (p -weak) upper gradients g of f ,

$$\int_B |f - f_B| d\mu \leq Cr \left(\int_{\lambda B} g^p d\mu \right)^{1/p},$$

where $f_B := \int_B f d\mu / \mu(B)$. If C and λ are independent of x , we say that X supports a (*global*) *p -Poincaré inequality*.

A nonnegative function w on \mathbf{R}^n is a *p -admissible weight* if $d\mu := w dx$ is globally doubling and \mathbf{R}^n equipped with μ supports a global p -Poincaré inequality. See Corollary 20.9 in [13] (which is only in the second edition) and Proposition A.17 in [2] for why this is equivalent to other definitions in the literature.

It is well known that if X supports a global p -Poincaré inequality, then X is connected, but in fact even a pointwise p -Poincaré inequality is sufficient for this.

Proposition 2.6. *If X supports a p -Poincaré inequality at x_0 , then X is connected and $C_p(S_R) > 0$ for every sphere $S_R = \{x : d(x, x_0) = R\}$ with $R < \text{diam } X/2$.*

In particular, if μ is globally doubling and X supports a global Poincaré inequality, then μ is reverse-doubling and $\tau > 1$ can be chosen arbitrarily.

Proof. The first part is shown in the same way as in Proposition 4.2 in [2]. For the second part, assume that $C_p(S_R) = 0$. Then 0 is a p -weak upper gradient of χ_{B_R} , as p -almost no curve intersects S_R , see [2, Proposition 1.48]. Thus the p -Poincaré inequality is violated for B_{2R} . \square

Definition 2.7. Let $\Omega \subset X$ be open. The *variational p -capacity* of $E \subset \Omega$ with respect to Ω is

$$\text{cap}_p(E, \Omega) = \inf_u \int_{\Omega} g_u^p d\mu,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $\chi_E \leq u \leq 1$ in E and $u = 0$ on $X \setminus E$; we call such functions u *admissible* for $\text{cap}_p(E, \Omega)$.

Also the variational capacity is countably subadditive and coincides with the usual variational capacity (see Björn–Björn [3, Theorem 5.1] for a proof valid in weighted \mathbf{R}^n).

Throughout the paper, we write $a \lesssim b$ if there is an implicit constant $C > 0$ such that $a \leq Cb$, where C is independent of the essential parameters involved. We also write $a \gtrsim b$ if $b \lesssim a$, and $a \simeq b$ if $a \lesssim b \lesssim a$.

Recall that $x_0 \in X$ is a fixed but arbitrary point and $B_r = B(x_0, r)$.

3. Upper bounds for capacity

In this section we prove Proposition 1.2 and show its sharpness.

Lemma 3.1. *If $0 < r < R$, then*

$$\text{cap}_p(B_r, B_R) \leq \frac{\mu(B_R \setminus B_r)}{(R-r)^p}.$$

Proof. The function

$$u(x) = \left(1 - \frac{\text{dist}(x, B_r)}{R-r}\right)_+$$

is admissible for $\text{cap}_p(B_r, B_R)$, and $g = (R-r)^{-1}\chi_{B_R \setminus B_r}$ is an upper gradient of u . We thus obtain that

$$\text{cap}_p(B_r, B_R) \leq \int_{B_R} g^p d\mu = \frac{\mu(B_R \setminus B_r)}{(R-r)^p}. \quad \square$$

Proof of Proposition 1.2. Using the η -AD property at x_0 and Lemma 3.1, we obtain that

$$\text{cap}_p(B_r, B_R) \leq \frac{\mu(B_R \setminus B_r)}{(R-r)^p} \lesssim \left(\frac{R-r}{R}\right)^\eta \frac{\mu(B_R)}{(R-r)^p} = \left(\frac{R-r}{R}\right)^{\eta-p} \frac{\mu(B_R)}{R^p}. \quad \square$$

Remark 3.2. Note that if μ has no AD property (in which case we could say that μ has the “0”-AD property), then Lemma 3.1 still gives

$$\text{cap}_p(B_r, B_R) \leq \left(1 - \frac{r}{R}\right)^{-p} \frac{\mu(B_R)}{R^p}.$$

This is sharp by Example 3.3 below.

Moreover, if μ has local η -AD at x_0 , in the sense that there is some $R_0 > 0$ such that (1.4) holds for all $0 < r < R < R_0$ as in Proposition 3.4 below, then (1.5) holds whenever $0 < r < R < R_0$. Similar local versions hold also for our other results.

It follows directly from the proof that the constant C from Definition 1.1 can be used as the implicit constant in (1.5).

The following example shows that Proposition 1.2 is sharp.

Example 3.3. (This example has been inspired by Example 1.3 in Buckley [7].) Let $x_0 = 0$, $0 < \eta < 1$ and $d\mu = w dx$ on \mathbf{R}^n , $n \geq 1$, where $w(x) = w(|x|)$ and

$$w(\rho) = \max\{1, |\rho - 1|^{\eta-1}\}.$$

This is a Muckenhoupt A_1 -weight, by Theorem II.3.4 in García-Cuerva–Rubio de Francia [10], and it is thus 1-admissible, by Theorem 4 in J. Björn [6]. It is easily verified that $\mu(B_R) \simeq R^n$ for all $R > 0$. We also see that $\mu(B_1 \setminus B_r) \simeq (1-r)^\eta$, if $\frac{1}{2} \leq r \leq 1$. One can check that this is the extreme case showing that the measure

μ has the global η -AD property (and that η is optimal). By Proposition 10.8 in Björn–Björn–Lehrbäck [5], for $p > 1$ and $\frac{1}{2} \leq r < 1$,

$$\begin{aligned} \text{cap}_{p,w}(B_r, B_1) &\simeq \left(\int_r^1 (w(\rho) \rho^{n-1})^{1/(1-p)} d\rho \right)^{1-p} \\ &\simeq \left(\int_r^1 (1-\rho)^{(\eta-1)/(1-p)} d\rho \right)^{1-p} \simeq (1-r)^{\eta-p}, \end{aligned} \quad (3.1)$$

which shows that the upper bound in Proposition 1.2 is sharp, with $R = 1$ fixed and $p > 1$.

Now let $d\tilde{\mu} = \tilde{w} dx$ and $d\mu_j = w_j dx$, where

$$\tilde{w}(\rho) := \sum_{j=0}^{\infty} a_j w_j(\rho), \quad \text{with } w_j(\rho) := w(q_j \rho), \quad j = 0, 1, \dots,$$

for some countable set $\{q_j\}_{j=0}^{\infty} \subset (0, \infty)$ (e.g. all positive rational numbers) and $a_j > 0$ such that $\sum_{j=0}^{\infty} a_j < \infty$. A change of variables shows that $\mu_j(B_R) = q_j^{-n} \mu(B_{q_j R}) \simeq R^n$ and hence $\tilde{\mu}(B_R) \simeq R^n$. Moreover, for $0 < r \leq R$ and $x \in X$,

$$\begin{aligned} \tilde{\mu}(B(x, R) \setminus B(x, r)) &= \sum_{j=0}^{\infty} a_j \mu_j(B(x, R) \setminus B(x, r)) \\ &= \sum_{j=0}^{\infty} a_j q_j^{-n} \mu(B(q_j x, q_j R) \setminus B(q_j x, q_j r)) \\ &\lesssim \sum_{j=0}^{\infty} a_j q_j^{-n} \left(1 - \frac{r}{R}\right)^{\eta} \mu(B(q_j x, q_j R)) \\ &= \left(1 - \frac{r}{R}\right)^{\eta} \tilde{\mu}(B(x, R)), \end{aligned}$$

i.e. $\tilde{\mu}$ has the global η -AD property as well. Since $\tilde{w} \geq a_j w_j$ for every $j = 0, 1, \dots$, we also see that η is optimal. Similarly, for every ball $B(x, r) \subset \mathbf{R}^n$, as w is an A_1 -weight,

$$\int_{B(x,r)} w_j dx = \int_{B(q_j x, q_j r)} w dx \lesssim \inf_{B(q_j x, q_j r)} w = \inf_{B(x,r)} w_j$$

and summing over all j shows that \tilde{w} is an A_1 -weight. Finally, using Proposition 10.8 in [5] again together with (3.1), we obtain for $p > 1$ and $\frac{1}{2} q_j^{-1} \leq r < R = q_j^{-1}$,

$$\begin{aligned} \text{cap}_{p,\tilde{w}}(B_r, B_R) &\simeq q_j^{1-n} \left(\int_r^R \left(\sum_{k=0}^{\infty} a_k w(q_k \rho) \right)^{1/(1-p)} d\rho \right)^{1-p} \\ &\gtrsim a_j q_j^{1-n} \left(q_j^{-1} \int_{q_j r}^1 w(\rho)^{1/(1-p)} d\rho \right)^{1-p} \\ &\simeq a_j q_j^{p-n} \text{cap}_{p,w}(B_{q_j r}, B_1) \\ &\simeq a_j \left(1 - \frac{r}{R}\right)^{\eta-p} \frac{\tilde{\mu}(B_R)}{R^p}, \end{aligned}$$

and letting $r \nearrow R$ shows that the upper bound in Proposition 1.2 is sharp for all $R = q_j^{-1}$.

For $p = 1$ we cannot use Proposition 10.8 in [5]. Instead we do as follows. Let $\frac{1}{2} q_j^{-1} \leq r < R = q_j^{-1}$, where $j = 0, 1, \dots$. Let u be admissible for $\text{cap}_{1,\tilde{w}}(B_r, B_R)$,

and let g be an upper gradient of u . We then get, using the unweighted capacity $\text{cap}_1(B_r, B_R)$ and (1.3),

$$\begin{aligned} \int_{\mathbf{R}^n} g \, d\tilde{\mu} &\geq \int_{\mathbf{S}^{n-1}} \int_r^R g(\rho\omega) \tilde{w}(\rho) \rho^{n-1} \, d\rho \, d\omega \\ &\gtrsim a_j w_j(r) \int_{\mathbf{S}^{n-1}} \int_r^R g(\rho\omega) \rho^{n-1} \, d\rho \, d\omega \\ &\geq a_j (1 - q_j r)^{\eta-1} \text{cap}_1(B_r, B_R) \\ &\simeq a_j \left(1 - \frac{r}{R}\right)^{\eta-1} \frac{\mu(B_R)}{R}. \end{aligned}$$

Taking infimum over all admissible u shows that Proposition 1.2 is sharp also for $p = 1$.

We have the following observation showing that the exponent $\eta = 1$ is the largest that can occur in the AD property, even locally, under a very mild assumption.

Proposition 3.4. *Let $x_0 \in X$ and $R_0 > 0$, and assume that $\mu(\{x_0\}) = 0$. If (1.4) holds for some $\eta > 0$ and all $0 < r < R < R_0$, then $\eta \leq 1$.*

Proof. Let $0 < R < R_0$. Using (1.4) we obtain for all integers $1 \leq k \leq K$,

$$\begin{aligned} \mu(B_R) &= \sum_{i=1}^K \mu(B_{iR/K} \setminus B_{(i-1)R/K}) \leq C \sum_{i=1}^K \left(1 - \frac{i-1}{i}\right)^{\eta} \mu(B_{iR/K}) \\ &\leq C \sum_{i=1}^k i^{-\eta} \mu(B_{kR/K}) + C \sum_{i=k+1}^{\infty} i^{-\eta} \mu(B_R), \end{aligned} \quad (3.2)$$

where C is the constant in (1.4). If $\eta > 1$, the series $\sum_{i=1}^{\infty} i^{-\eta}$ converges and we can find k such that $C \sum_{i=k+1}^{\infty} i^{-\eta} \leq \frac{1}{2}$. Thus, subtracting the last term in (3.2) from the left-hand side yields

$$\mu(B_R) \lesssim \mu(B_{kR/K}) \rightarrow 0, \quad \text{as } K \rightarrow \infty,$$

which is impossible. Thus $\eta \leq 1$. \square

4. Lower bounds for capacity

We now turn to lower estimates for capacities of thin annuli. The following is our main estimate for obtaining the lower bound in Theorem 1.3. As usual for lower bounds, a key assumption is some sort of a Poincaré inequality.

Theorem 4.1. *Assume that $1 \leq q < p < \infty$, that X supports a q -Poincaré inequality at x_0 , and that μ has the η -AD property at x_0 and is reverse-doubling at x_0 with dilation $\tau > 1$. Then*

$$\text{cap}_p(B_r, B_R) \gtrsim \left(1 - \frac{r}{R}\right)^{\eta(q-p)/q} \frac{\mu(B_R)}{R^p}, \quad \text{if } 0 < \frac{R}{2} \leq r < R \leq \frac{\text{diam } X}{2\tau}. \quad (4.1)$$

If μ has the global η -AD property and supports a global q -Poincaré inequality, then the implicit constants are independent of x_0 and we may choose $\tau > 1$ arbitrarily, see the proof of Theorem 1.3 below.

Example 7.3 shows that the reverse-doubling assumption cannot be dropped, while Example 7.1 shows that it is not enough to assume that X supports global q -Poincaré inequalities for all $q > p$. Moreover, Example 7.2 shows that the η -AD property cannot be replaced by the assumption that μ is globally doubling or even Ahlfors regular.

Proof. Let u be admissible for $\text{cap}_p(B_r, B_R)$. Then $u = 1$ in B_r , $u = 0$ outside B_R , and $g_u = 0$ a.e. outside $B_R \setminus B_r$. The reverse-doubling implies that $\mu(B_{\tau R} \setminus B_R) \gtrsim \mu(B_R)$ from which it follows that $|u_{B_{\tau R}}| < c < 1$, and so $|u - u_{B_{\tau R}}| > 1 - c > 0$ in $B_{R/2}$. Note that the η -AD property implies that μ is doubling at x_0 . Thus we obtain from the q -Poincaré inequality at x_0 and Hölder's inequality that

$$\begin{aligned} 1 &\lesssim \int_{B_{R/2}} |u - u_{B_{\tau R}}| d\mu \lesssim \int_{B_{\tau R}} |u - u_{B_{\tau R}}| d\mu \lesssim R \left(\int_{B_{\tau \lambda R}} g_u^q d\mu \right)^{1/q} \\ &\lesssim \frac{R}{\mu(B_R)^{1/q}} \left(\int_{B_R \setminus B_r} g_u^q d\mu \right)^{1/q} \\ &\lesssim \frac{R}{\mu(B_R)^{1/q}} \mu(B_R \setminus B_r)^{1/q-1/p} \left(\int_{B_R \setminus B_r} g_u^p d\mu \right)^{1/p}. \end{aligned}$$

By the η -AD property, $\mu(B_R \setminus B_r) \lesssim (1 - r/R)^\eta \mu(B_R)$. Inserting this into the above estimate yields

$$\begin{aligned} \left(\int_{B_R \setminus B_r} g_u^p d\mu \right)^{1/p} &\gtrsim \frac{\mu(B_R)^{1/q}}{R} \left(1 - \frac{r}{R} \right)^{\eta(1/p-1/q)} \mu(B_R)^{1/p-1/q} \\ &= \frac{\mu(B_R)^{1/p}}{R} \left(1 - \frac{r}{R} \right)^{\eta(q-p)/pq}, \end{aligned}$$

and (4.1) follows after taking infimum over all admissible u . \square

Theorem 4.1 establishes the lower bound in Theorem 1.3 when $p > 1$. For $p = 1$ we instead use the following result. In view of Remark 3.2, we can see this as an $\eta = 0$ version of Theorem 4.1.

Proposition 4.2. *Assume that X supports a p -Poincaré inequality at x_0 , and that μ is doubling at x_0 and reverse-doubling at x_0 with dilation $\tau > 1$. Then*

$$\text{cap}_p(B_r, B_R) \gtrsim \frac{\mu(B_R)}{R^p}, \quad \text{if } 0 < \frac{R}{2} \leq r < R \leq \frac{\text{diam } X}{2\tau}. \quad (4.2)$$

Moreover, $\text{cap}_p(B_r, B_{2r}) \simeq \mu(B_r) r^{-p}$ when $0 < r \leq \text{diam } X / 4\tau$.

If μ is globally doubling and X supports a global p -Poincaré inequality, then the implicit constants are independent of x_0 .

Example 7.2 shows that the lower estimate is sharp even under the assumptions that μ is globally doubling (or Ahlfors regular) and X supports a global 1-Poincaré inequality. Example 7.1 shows that the p -Poincaré assumption cannot be weakened, even if it is assumed globally. Example 7.4 shows that the doubling assumption cannot be dropped (not even if X supports a global 1-Poincaré inequality and μ is globally reverse-doubling), while Example 7.3 shows that the reverse-doubling assumption cannot be dropped. See also Proposition 7.5.

Proof. Let u be admissible for $\text{cap}_p(B_r, B_R)$. As in the proof of Theorem 4.1 (with q replaced by p), we get that

$$1 \lesssim R \left(\int_{B_{\tau \lambda R}} g_u^p d\mu \right)^{1/p} \lesssim \frac{R}{\mu(B_R)^{1/p}} \left(\int_{B_R \setminus B_r} g_u^p d\mu \right)^{1/p},$$

and (4.2) follows after taking infimum over all admissible u . That $\text{cap}_p(B_r, B_{2r}) \simeq \mu(B_r) r^{-p}$ follows from this and Lemma 3.1. \square

Theorem 4.3. *Assume that X supports a 1-Poincaré inequality at x_0 and that μ has the 1-AD property at x_0 and is reverse-doubling at x_0 with dilation $\tau > 1$. Then*

$$\text{cap}_p(B_r, B_R) \simeq \left(1 - \frac{r}{R}\right)^{1-p} \frac{\mu(B_R)}{R^p}, \quad \text{if } 0 < \frac{R}{2} \leq r < R \leq \frac{\text{diam } X}{2\tau}.$$

Even under global assumptions, as in Theorem 1.3, the 1-Poincaré and 1-AD assumptions cannot be weakened, as shown by Example 7.1. Example 7.3 shows that the reverse-doubling assumption cannot be dropped, and in particular that it is possible that μ has the 1-AD property at x_0 and X supports a 1-Poincaré inequality at x_0 , but that μ fails to be reverse-doubling at x_0 .

Proof. This follows by combining Proposition 1.2 with Theorem 4.1 (for $p > 1$) and Proposition 4.2 (for $p = 1$). \square

Proof of Theorem 1.3. It follows from the global assumptions and Proposition 2.6 that X is connected. Hence, X is reverse-doubling at x_0 with $\tau = \frac{3}{2}$. As the implicit constants in Theorem 4.3 only depend on the parameters in the assumptions, Theorem 1.3 follows. \square

The following result gives a two-sided estimate of a different form.

Theorem 4.4. *Assume that μ is globally doubling and that X supports a global p -Poincaré inequality. Let $0 < \frac{1}{2}R \leq r < R$ and $\delta = R - r$. Assume, in addition, that there exists $a > 0$ such that for every $x \in B_R \setminus B_r$ there exist x' and x'' so that $B(x', a\delta) \subset B(x, 2\delta) \cap B_r$ and $B(x'', a\delta) \subset B(x, 2\delta) \setminus B_R$. Then*

$$\text{cap}_p(B_r, B_R) \simeq \frac{\mu(B_R \setminus B_r)}{(R - r)^p}.$$

The balls $B(x', a\delta)$ in the assumptions of Theorem 4.4 always exist e.g. if X is a length space. The existence of the balls $B(x'', a\delta)$ is more difficult to guarantee but there are plenty of spaces where it is true. For example, \mathbf{R}^n equipped with any p -admissible measure satisfies the assumptions.

Observe that the geometric condition is only assumed to hold for the specific r and R under consideration, but whenever the geometric condition is satisfied the implicit constants in the estimate are independent of r and R . Clearly, the constant 2 in $B(x, 2\delta)$ is not important and can be replaced by any number ≥ 2 . This may be useful for some spaces containing well distributed holes. The same is true for Corollary 4.5 below.

That the geometric assumption cannot be dropped is shown by Example 7.2, while Example 7.1 shows that the Poincaré assumption cannot be weakened. Examples 7.3 and 7.4 show that the assumption that μ is globally doubling can neither be replaced by the assumption that μ is doubling at x_0 , nor by the assumption that μ is globally reverse-doubling, i.e. reverse doubling at every $x \in X$ with uniform constants.

Proof. The upper bound follows from Lemma 3.1, so it suffices to prove the lower bound.

Use the Hausdorff maximality principle to find a maximal pairwise disjoint collection of balls $B(x_j, \delta)$ with $x_j \in B_R \setminus B_r$. By maximality, the balls $B_j = B(x_j, 2\delta)$ cover $B_R \setminus B_r$. Moreover, since μ is globally doubling, it can be shown that the balls λB_j have bounded overlap depending only on λ and the doubling constant of μ . Now, for each j let $B'_j = B(x'_j, a\delta)$ and $B''_j = B(x''_j, a\delta)$ as in the assumption of the theorem.

Let u be admissible for $\text{cap}_p(B_r, B_R)$. Then $u = 1$ in B_r and $u = 0$ outside B_R . In particular, $u = 1$ in each B'_j and $u = 0$ in each B''_j . Since μ is globally doubling,

it follows that $u_{B_j} \leq 1 - \mu(B_j'')/\mu(B_j) \leq c$, where $c < 1$ is independent of j . An application of the global p -Poincaré inequality to B_j , and using that $g_u = 0$ a.e. outside $B_R \setminus B_r$, then yields

$$\begin{aligned} 0 < 1 - c &\leq \int_{B_j'} |u - u_{B_j}| d\mu \lesssim \int_{B_j} |u - u_{B_j}| d\mu \\ &\lesssim \delta \left(\frac{1}{\mu(\lambda B_j)} \int_{\lambda B_j} g_u^p d\mu \right)^{1/p} = \delta \left(\frac{1}{\mu(\lambda B_j)} \int_{\lambda B_j \cap (B_R \setminus B_r)} g_u^p d\mu \right)^{1/p}. \end{aligned}$$

From this it follows that

$$\mu(\lambda B_j \cap (B_R \setminus B_r)) \leq \mu(\lambda B_j) \lesssim \delta^p \int_{\lambda B_j \cap (B_R \setminus B_r)} g_u^p d\mu.$$

Since the balls λB_j have bounded overlap and cover $B_R \setminus B_r$, summing over all j gives

$$\mu(B_R \setminus B_r) \lesssim \delta^p \int_{B_R \setminus B_r} g_u^p d\mu$$

and taking infimum over all admissible u proves the lower bound. \square

The following corollary partly complements Theorem 4.1 and the lower bound in Theorem 4.3 in the case when the 1-AD property or a 1-Poincaré inequality are not satisfied. In particular, if the doubling condition and a 1-Poincaré inequality hold globally and $p > 1$, then the 1-AD condition can be replaced by the geometric condition in Theorem 4.4. That the geometric assumption cannot be dropped is shown by Example 7.2, while Example 7.1 shows that the pointwise Poincaré assumption cannot be weakened.

Corollary 4.5. *If the assumptions of Theorem 4.4 are satisfied and in addition X supports a q -Poincaré inequality at x_0 for some $1 \leq q < p$, then*

$$\text{cap}_p(B_r, B_R) \gtrsim \left(1 - \frac{r}{R}\right)^{q-p} \frac{\mu(B_R)}{R^p}.$$

Proof. This follows directly from Theorem 4.4 and the following Lemma 4.6. \square

The following estimate complements the 1-AD property. In particular, if $q = 1$ then this lower bound, together with the 1-AD property, leads to a sharp two-sided estimate for $\mu(B_R \setminus B_r)$ when r is close to R .

Lemma 4.6. *Assume that X supports a q -Poincaré inequality at x_0 for some $1 \leq q < \infty$ and that μ is doubling at x_0 and reverse-doubling at x_0 . Then*

$$\mu(B_R \setminus B_r) \gtrsim \left(1 - \frac{r}{R}\right)^q \mu(B_R) \quad \text{when } 0 < \frac{R}{2} \leq r < R < \frac{\text{diam } X}{2\tau}.$$

Example 7.1 shows that the Poincaré assumption cannot be weakened, while Example 7.3 shows that the reverse-doubling condition cannot be omitted. We do not know if the doubling condition can be omitted.

Proof. Let

$$u(x) = \left(1 - \frac{\text{dist}(x, B_r)}{R - r}\right)_+.$$

As in the proof of Lemma 3.1, we obtain that

$$\int_X g_u^q d\mu \leq \frac{\mu(B_R \setminus B_r)}{(R - r)^q}.$$

On the other hand, as in the proof of Theorem 4.1, we get that

$$1 \lesssim R \left(\int_{B_{\tau\lambda R}} g_u^q d\mu \right)^{1/q} \lesssim \frac{R}{\mu(B_R)^{1/q}} \frac{\mu(B_R \setminus B_r)^{1/q}}{(R-r)},$$

and the claim follows. \square

We can also obtain the following variant of Corollary 4.5.

Proposition 4.7. *If the assumptions of Theorem 4.4 are satisfied with $p > 1$, then there is $1 \leq q < p$ such that*

$$\text{cap}_p(B_r, B_R) \gtrsim \left(1 - \frac{r}{R}\right)^{q-p} \frac{\mu(B_R)}{R^p}.$$

As seen from the proof below (and those in [1] and [16]) q only depends on p and the constants in the global doubling condition and the global p -Poincaré inequality. Moreover, it also follows from the proof that the completion \widehat{X} of X supports a global q -Poincaré inequality for this q . In fact, it would be enough to require that \widehat{X} supports a q -Poincaré inequality at x_0 . Note that Koskela [17, Theorems A and C] has given counterexamples showing that X may not support any better Poincaré inequality than the p -Poincaré inequality (so Corollary 4.5 is not at our disposal). His examples are of the type $X = \mathbf{R}^n \setminus E$, where $E \subset \mathbf{R}^{n-1}$ so they satisfy the geometric condition in Theorem 4.4 and \widehat{X} supports a global 1-Poincaré inequality.

Proof. Let \widehat{X} be the completion of X and extend the measure μ so that $\mu(\widehat{X} \setminus X) = 0$. Then μ is doubling on \widehat{X} and \widehat{X} supports a p -Poincaré inequality, by Proposition 7.1 in Aikawa–Shanmugalingam [1]. By Theorem 1.0.1 in Keith–Zhong [16], it follows that there is $1 \leq q < p$ such that \widehat{X} supports a q -Poincaré inequality. Now we can apply Lemma 4.6 with respect to \widehat{X} , and since the estimate of Lemma 4.6 holds for the measure μ also when restricted to X , this estimate together with Theorem 4.4 completes the proof. \square

5. The blowup of $\text{cap}_p(B_r, B_R)$ as $r \rightarrow R$

Theorem 4.1, Corollary 4.5 and Proposition 4.7 all give uniform estimates for the blowup of $\text{cap}_p(B_r, B_R)$ as $r \rightarrow R$. In particular they show that

$$\lim_{\delta \rightarrow 0+} \text{cap}_p(B_{R-\delta}, B_R) = \lim_{\delta \rightarrow 0+} \text{cap}_p(B_R, B_{R+\delta}) = \infty$$

when the respective assumptions are satisfied.

If we are not interested in uniform estimates, but only in the limits above, then these can be obtained under considerably weaker assumptions, as we will now show.

Proposition 5.1. *Assume that $1 \leq q < p < \infty$ and that X supports a q -Poincaré inequality at x_0 . Let $R > 0$ be such that $\mu(X \setminus B_R) > 0$, which in particular holds if $X \setminus \overline{B}_R \neq \emptyset$. Then*

$$\lim_{\delta \rightarrow 0+} \text{cap}_p(B_{R-\delta}, B_R) = \infty. \quad (5.1)$$

If in addition $\mu(\{y : d(y, x_0) = R\}) = 0$, then also

$$\lim_{\delta \rightarrow 0+} \text{cap}_p(B_R, B_{R+\delta}) = \infty. \quad (5.2)$$

If $X = B_R$ then $\text{cap}_p(B_{R-\delta}, B_R) = \text{cap}_p(B_R, B_{R+\delta}) = 0$, and thus the condition $\mu(X \setminus B_R) > 0$ cannot be dropped for either limit. Example 7.1 shows that it is not enough to assume that X supports global q -Poincaré inequalities for all $q > p$ for neither limit, but we do not know if it is enough to assume that X supports a p -Poincaré inequality at x_0 . Moreover, Example 7.2 shows that the assumption $\mu(\{y : d(y, x_0) = R\}) = 0$ cannot be dropped for the limit (5.2) to hold, even if X supports a global 1-Poincaré inequality. If $p = 1$ the result fails even if we assume a global 1-Poincaré inequality, as seen by considering \mathbf{R}^n or Theorem 1.3.

Proof. Assume that $0 < \delta < \frac{1}{2}R$ and let $r = R - \delta$. Let u be admissible for $\text{cap}_p(B_r, B_R)$. Then, following the ideas in the proof of Theorem 4.1,

$$\begin{aligned} 1 - \frac{\mu(B_R)}{\mu(B_{2R})} &\leq \int_{B_{R/2}} |u - u_{B_{2R}}| d\mu \\ &\leq \frac{\mu(B_{2R})}{\mu(B_{R/2})} \int_{B_{2R}} |u - u_{B_{2R}}| d\mu \\ &\leq CR \frac{\mu(B_{2R})}{\mu(B_{R/2})} \left(\int_{B_{2\lambda R}} g_u^q d\mu \right)^{1/q} \\ &= \frac{CR\mu(B_{2R})}{\mu(B_{R/2})\mu(B_{2\lambda R})^{1/q}} \left(\int_{B_R \setminus B_r} g_u^q d\mu \right)^{1/q} \\ &\leq \frac{CR\mu(B_{2R})}{\mu(B_{R/2})\mu(B_{2\lambda R})^{1/q}} \mu(B_R \setminus B_r)^{1/q-1/p} \left(\int_{B_R \setminus B_r} g_u^p d\mu \right)^{1/p}. \end{aligned}$$

Taking infimum over all admissible u shows that

$$\text{cap}_p(B_r, B_R) \geq \left(1 - \frac{\mu(B_R)}{\mu(B_{2R})} \right)^p \left(\frac{\mu(B_{R/2})\mu(B_{2\lambda R})^{1/q}}{CR\mu(B_{2R})} \right)^p \mu(B_R \setminus B_r)^{1-p/q}. \quad (5.3)$$

To see that the first factor in the right-hand side is positive, we note that either $X = \{y : d(y, x_0) \leq R\}$ or there is a point y with $R < d(y, x_0) < \frac{3}{2}R$, as X is connected by Proposition 2.6. In the former case, $\mu(B_{2R} \setminus B_R) = \mu(X \setminus B_R) > 0$, while in the latter case $\mu(B_{2R} \setminus B_R) \geq \mu(B(y, d(y, x_0) - R)) > 0$. Thus the first factor in (5.3) is positive, and so is clearly the second one as well. Since the last factor tends to ∞ , as $\delta \rightarrow 0+$, we see that (5.1) holds.

The proof of (5.2) is similar (one can also use (5.3) directly), but in this case one needs to use that $\mu(B_{R+\delta} \setminus B_R) \rightarrow \mu(\{y : d(y, x_0) = R\}) = 0$, as $\delta \rightarrow 0+$. \square

6. Characterizations of the 1-AD property

Our aim in this section is to characterize the 1-AD property.

Theorem 6.1. *Let $f(r) := \mu(B_r)$. Then the following are equivalent:*

- (a) μ has the 1-AD property at x_0 ;
- (b) f is locally absolutely continuous on $(0, \infty)$ and $\rho f'(\rho) \lesssim f(\rho)$ for a.e. $\rho > 0$;
- (c) f is locally Lipschitz on $(0, \infty)$ and $\rho f'(\rho) \lesssim f(\rho)$ for a.e. $\rho > 0$.

If moreover μ is reverse-doubling at x_0 and X supports a 1-Poincaré inequality at x_0 , then also the following condition is equivalent to those above:

- (d) f is locally Lipschitz on $(0, \infty)$ and $\rho f'(\rho) \simeq f(\rho)$ for a.e. ρ with $0 < \rho < \text{diam } X/2\tau$.

The assumption of absolute continuity cannot be dropped, as shown by Example 2.6 in Björn–Björn–Lehrbäck [5] where X is the usual Cantor ternary set and f is the Cantor staircase function for which $f'(\rho) = 0 \leq f(\rho)/\rho$ for a.e. $\rho > 0$.

Example 7.1 shows that the 1-Poincaré assumption (for the last part) cannot be weakened, even under global assumptions, while Example 7.3 shows that the reverse-doubling assumption cannot be dropped even if X supports a global 1-Poincaré inequality.

Proof. (a) \Rightarrow (c) By the 1-AD property at x_0 we have for $0 < \varepsilon < \rho$ that

$$\frac{f(\rho) - f(\rho - \varepsilon)}{\varepsilon} \lesssim \frac{\left(1 - \frac{\rho - \varepsilon}{\rho}\right) f(\rho)}{\varepsilon} = \frac{f(\rho)}{\rho}. \quad (6.1)$$

Since the right-hand side is locally bounded it follows that f is locally Lipschitz on $(0, \infty)$, and thus that $f'(\rho)$ exists for a.e. $\rho > 0$. Moreover, by (6.1) we see that $f'(\rho) \lesssim f(\rho)/\rho$ whenever $f'(\rho)$ exists.

(c) \Rightarrow (b) This is trivial.

(b) \Rightarrow (a) Assume that $\rho f'(\rho)/f(\rho) \leq M$ a.e. We have

$$\frac{\mu(B_R \setminus B_r)}{\mu(B_R)} = 1 - \frac{f(r)}{f(R)} = 1 - \exp(h(r) - h(R)), \quad (6.2)$$

where $h(\rho) = \log f(\rho)$ is also locally absolutely continuous with $h'(\rho) = f'(\rho)/f(\rho) \leq M/\rho$ for a.e. $\rho > 0$. It follows that

$$h(r) - h(R) = - \int_r^R h'(\rho) d\rho \geq -M \int_r^R \frac{d\rho}{\rho} = \log\left(\frac{r}{R}\right)^M.$$

Inserting this into (6.2) yields

$$1 - \frac{f(r)}{f(R)} \leq 1 - \left(\frac{r}{R}\right)^M.$$

Finally, Lemma 3.1 from Björn–Björn–Gill–Shanmugalingam [4] shows that for $t \in [0, 1]$,

$$\min\{1, M\}t \leq 1 - (1 - t)^M \leq \max\{1, M\}t,$$

and applying this with $t = 1 - r/R$ concludes the proof.

Thus we have shown that (a)–(c) are equivalent.

Now assume that μ is reverse-doubling at x_0 , and that X supports a 1-Poincaré inequality at x_0 .

(c) \Rightarrow (d) Let $0 < \rho < \frac{1}{3} \text{diam } X$ and $0 < \varepsilon < \frac{1}{2}\rho$. We have already shown that (c) \Rightarrow (a), so μ has the 1-AD property at x_0 , and in particular μ is doubling at x_0 . Thus Lemma 4.6 (with $q = 1$) yields

$$\frac{f(\rho) - f(\rho - \varepsilon)}{\varepsilon} \gtrsim \frac{\left(1 - \frac{\rho - \varepsilon}{\rho}\right) f(\rho)}{\varepsilon} = \frac{f(\rho)}{\rho},$$

showing that $f'(\rho) \gtrsim f(\rho)/\rho$ whenever $f'(\rho)$ exists.

(d) \Rightarrow (c) If X is unbounded this is trivial. So assume that X is bounded. If $\rho > \text{diam } X$, then $f(\rho) = \mu(X)$ and $f'(\rho) = 0$. As f is locally Lipschitz there is a constant M such that $f'(\rho) \leq M$ for a.e. ρ satisfying $\frac{1}{3} \text{diam } X < \rho < \text{diam } X$. For such ρ we have that $f(\rho)/\rho \geq f(B_{\text{diam } X/3})/\text{diam } X$ and thus $\rho f'(\rho) \lesssim f(\rho)$ for a.e. $\rho > \frac{1}{3} \text{diam } X$. Together with (d) this yields (c). \square

In \mathbf{R}^n , the measure of a ball can be obtained by one-dimensional integration of the surface measures of spheres. To do the same in metric spaces we need the following lemma, which is also useful for verifying the conditions in Theorem 6.1.

Lemma 6.2. *Assume that μ is globally doubling and that X supports a global q -Poincaré inequality for some $1 \leq q < \infty$. Assume, in addition, that there exists $a > 0$ such that whenever $0 < r < R \leq 2r$ and $\delta = R - r$, for every $x \in B_R \setminus B_r$ there exist x' and x'' so that $B(x', a\delta) \subset B(x, 2\delta) \cap B_r$ and $B(x'', a\delta) \subset B(x, 2\delta) \setminus B_R$.*

Then the function $f(r) := \mu(B_r)$ is locally absolutely continuous on $(0, \infty)$.

The values of the constants a and 2 (in $B(x, 2\delta)$) are not important, and by covering $(0, \infty)$ we can even allow them to be different in different parts. Thus, we can replace the last assumption by the following condition: for each $k \in \mathbf{Z}$ there exist $a_k > 0$ and $b_k \geq 2$ such that whenever $4^k < r < R \leq 2r < 4^{k+2}$ and $\delta = R - r$, for every $x \in B_R \setminus B_r$ there exist x' and x'' so that $B(x', a_k\delta) \subset B(x, b_k\delta) \cap B_r$ and $B(x'', a_k\delta) \subset B(x, b_k\delta) \setminus B_R$.

Proof. It suffices to show that the measure ν defined on $(0, \infty)$ by

$$\nu(E) = \mu(\{x \in X : d(x_0, x) \in E\})$$

is absolutely continuous with respect to the Lebesgue measure.

Let $0 < r < R \leq 2r$, $\delta = R - r$ and $I = (r, R)$. We start by showing that for all measurable $E \subset I$,

$$\left(\frac{|E|}{|I|}\right)^q \lesssim \frac{\nu(E)}{\nu(I)}, \quad (6.3)$$

where q is the exponent from the assumed global q -Poincaré inequality. To this end, set for $t > 0$,

$$u(x) = |E| - \int_0^{d(x_0, x)} \chi_E(\tau) d\tau, \quad x \in X,$$

and note that $u = |E|$ in B_r , $u = 0$ outside B_R and $g_u(x) \leq \chi_E(d(x_0, x))$ a.e.

Let the balls B_j , B'_j and B''_j be as in the proof of Theorem 4.4. Hence, in the same way as in the proof of Theorem 4.4, we obtain

$$(1 - c)|E| \lesssim \delta \left(\int_{\lambda B_j} g_u^q d\mu \right)^{1/q} \leq |I| \left(\frac{\mu(\{x \in \lambda B_j : d(x_0, x) \in E\})}{\mu(\lambda B_j \cap (B_R \setminus B_r))} \right)^{1/q},$$

or equivalently,

$$|E|^q \mu(\lambda B_j \cap (B_R \setminus B_r)) \lesssim |I|^q \mu(\{x \in \lambda B_j : d(x_0, x) \in E\}).$$

Since the balls λB_j cover the annulus $B_R \setminus B_r$ and have bounded overlap, summing over all j gives (6.3).

Now assume for a contradiction that there exists $E \subset (r, R)$ such that $|E| = 0$ and $\nu(E) > 0$. As ν is a Radon measure on $(0, \infty)$, the Lebesgue differentiation theorem holds with respect to ν , see Remark 1.13 in Heinonen [12]. Thus, there exists at least one $x \in E$ which is a Lebesgue point with respect to ν of the function $1 - \chi_E$. Hence, for every $\varepsilon > 0$, there is an interval I_ε such that $x \in I_\varepsilon \subset I$ and $\nu(I_\varepsilon \setminus E) < \varepsilon \nu(I_\varepsilon)$. Applying (6.3) to $I_\varepsilon \setminus E$ and I_ε in place of E and I gives

$$1 = \left(\frac{|I_\varepsilon \setminus E|}{|I_\varepsilon|} \right)^q \lesssim \frac{\nu(I_\varepsilon \setminus E)}{\nu(I_\varepsilon)} < \varepsilon,$$

which is impossible. Thus, the assumption that $\nu(E) > 0$ was false and we have shown that ν is absolutely continuous with respect to the Lebesgue measure on every interval (r, R) , and hence on $(0, \infty)$. \square

The following result is now a direct consequence of Theorem 6.1 and Lemma 6.2.

Corollary 6.3. *Under the assumptions of Lemma 6.2, μ has the 1-AD property at x_0 if and only if the function $f(r) := \mu(B_r)$ satisfies $\rho f'(\rho) \lesssim f(\rho)$ for a.e. $\rho > 0$.*

7. Counterexamples

In this section we provide a number of counterexamples showing that most of our results are sharp. The following example shows the sharpness both of the Poincaré and AD assumptions in Theorem 1.3. It also shows sharpness of the Poincaré assumptions in several other results.

Example 7.1. (Weighted bow-tie) Let

$$X = \{(x_1, \dots, x_n) : x_2^2 + \dots + x_n^2 \leq \frac{1}{4}x_1^2 \text{ and } -1 \leq x_1 \leq 2\} \quad (7.1)$$

as a subset of \mathbf{R}^n , $n \geq 2$, and equip X with the measure $d\mu = |x|^\alpha dx$, where $\alpha > -n$. (Additionally, we can make this example into a length space if we equip X with the inner metric (see [2, Definition 4.41]), which only makes a difference when calculating distances between the two sides of the origin.) Note that the constant 2 in the range of x_1 in (7.1) above was chosen so that we can have $R = 1 \leq \frac{1}{3} \text{diam } X$ below as required in Theorem 1.3.

If $q \geq 1$, then X supports a global q -Poincaré inequality if and only if $q > n + \alpha$ or $q = 1 \geq n + \alpha$, see Example 5.7 in [2]. Moreover, μ is globally doubling.

Let $x_0 = (-1, 0, \dots, 0)$ and $\eta = \min\{1, n + \alpha\}$. Then for $0 < r < R < \text{diam } X$ we have

$$\mu(B_R) \simeq R^n \quad \text{and} \quad \mu(B_R \setminus B_r) \lesssim \left(1 - \frac{r}{R}\right)^\eta \mu(B_R),$$

which shows that μ has the η -AD property at x_0 . One can check that this is the extreme case showing that μ has the global η -AD property (and that η is optimal).

If $0 < \delta < \frac{1}{2}$, then

$$\mu(B_1 \setminus B_{1-\delta}) \simeq \int_0^\delta \rho^\alpha \rho^{n-1} d\rho \simeq \delta^{n+\alpha},$$

which shows that the Poincaré assumption in Lemma 4.6 cannot be weakened. Moreover, by Lemma 3.1,

$$\text{cap}_p(B_{1-\delta}, B_1) \lesssim \frac{1}{\delta^p} \mu(B_1 \setminus B_{1-\delta}) \simeq \delta^{n+\alpha-p},$$

which shows that (1.6) fails if $n + \alpha > 1$, and thus we cannot replace the assumption of a global 1-Poincaré inequality in Theorem 1.3 by a global q -Poincaré inequality for any fixed $q > 1$. Nor can the pointwise q -Poincaré inequality in Corollary 4.5 be replaced by assuming that X supports a global q' -Poincaré inequality for any fixed $q' > q$.

Conversely, if $0 < \delta < \frac{1}{2}$ and X supports a global p -Poincaré inequality, i.e. if $p > n + \alpha$, then a simple reflection argument and [5, Proposition 10.8] imply that

$$\text{cap}_p(B_{1-\delta}, B_1) \gtrsim \text{cap}_p(\{0\}, B(0, 2\delta)) \simeq \delta^{n+\alpha-p}$$

and hence

$$\text{cap}_p(B_{1-\delta}, B_1) \simeq \delta^{n+\alpha-p}.$$

If $\eta = n + \alpha < 1$, then X supports a global 1-Poincaré inequality and μ has the η -AD property at x_0 , but (1.6) fails. Hence we cannot replace the 1-AD assumption in Theorem 1.3 by the η -AD property for any fixed $\eta < 1$. In fact, it is only the upper bound in (1.6) that fails. The lower bound therein is still provided by Corollary 4.5.

Next, if $p = n + \alpha > 1$, then X supports a global q -Poincaré inequality for each $q > p$, but not a global p -Poincaré inequality. Moreover, by Example 5.7 in [2], $C_p(\{0\}) = 0$ and thus we can test $\text{cap}_p(B_{1-\delta}, B_1)$ with $u = \chi_{B_1}$ yielding $\text{cap}_p(B_{1-\delta}, B_1) = 0$. It also follows from Proposition 2.6 that X does not support

a p -Poincaré inequality at x_0 . Hence the p -Poincaré assumption in Proposition 4.2 cannot be weakened if $p > 1$. Moreover, it also follows that it is not enough to assume that X supports global q -Poincaré inequalities for all $q > p$ in Theorems 4.1 and 4.4 when $p > 1$, as well as for (5.1) in Proposition 5.1 to hold. As in this case we also have $\text{cap}_p(B_1, B_{1+\delta}) = 0$, the same is true for (5.2) in Proposition 5.1.

When $p = 1 < q$ we instead choose n and α so that $1 < n + \alpha < q$. In particular, X supports a global q -Poincaré inequality in this case. As above, $\text{cap}_1(B_{1-\delta}, B_1) = 0$ and X does not support a 1-Poincaré inequality at x_0 , showing that the Poincaré assumption in Proposition 4.2 is sharp also for $p = 1$. It also follows that when $p = 1$ it is not enough to assume that X supports a global q -Poincaré inequality for some fixed $q > 1$ in Theorem 4.4.

Let now, as in Theorem 6.1, $f(r) = \mu(B_r)$. Let $q > 1$ and choose n and α so that $1 < n + \alpha < q$. Then μ has the global 1-AD property and X supports a global q -Poincaré inequality. For $\frac{1}{2} \leq r < R \leq 1$, with R close to r , we see that

$$\mu(B_R \setminus B_r) \simeq (1-r)^\alpha m(B_R \setminus B_r) \simeq (1-r)^\alpha (R-r)(1-r)^{n-1}$$

where m is the n -dimensional Lebesgue measure. Hence

$$rf'(r) = r \lim_{R \rightarrow r+} \frac{\mu(B_R \setminus B_r)}{R-r} \simeq r(1-r)^{n+\alpha-1} \not\simeq r \simeq r^n \simeq \mu(B_r) \quad \text{when } \frac{1}{2} < r < 1.$$

Thus condition (d) in Theorem 6.1 fails, which shows that it is not enough to assume that X supports a global q -Poincaré inequality for some fixed $q > 1$ in (the last part of) Theorem 6.1.

We do not have a counterexample to the conclusion of Theorem 4.3 which supports pointwise q -Poincaré inequalities at x_0 for all $q > 1$.

Example 7.2. (This example was introduced by Tessera [22, p. 50] in a different context. See also Routin [19, Section 6] for a more detailed discussion of this space.) Let X consist of the intervals (with the natural embedding of \mathbf{R} into \mathbf{R}^2) $[0, 1]$, $[2^{k-1}, 2^k]$ for even positive k and $[-2^k, -2^{k-1}]$ for odd positive k , and of the half-circles centred at the origin and of radius 2^k lying in the upper half-plane for even nonnegative k and in the lower half-plane for odd positive k . We equip X with the Euclidean metric d inherited from \mathbf{R}^2 and the 1-dimensional Hausdorff measure μ . Then X is Ahlfors 1-regular (see Proposition 6.1 in [19]). Moreover, X is bi-Lipschitz equivalent to the half-line $[0, \infty) \subset \mathbf{R}$, and hence supports a global 1-Poincaré inequality (by [2, Proposition 4.16]).

Let $x_0 = 0$, $k > 0$ be an integer, $\delta > 0$ be small, $r = 2^k - \delta$ and $R = 2^k + \delta$. Then $\mu(B_R \setminus B_r) \simeq R \simeq \mu(B_R)$, which shows that μ does not have the η -AD property at x_0 for any $\eta > 0$.

Considering the function u which is 1 when $|x| < 2^k$ and 0 when $|x| > 2^k$, and decays linearly from 1 to 0 along the half-circle of radius 2^k , it is easy to see that for the above balls $\text{cap}_p(B_r, B_R) \lesssim R^{1-p} \simeq \mu(B_R)R^{-p}$. Together with Proposition 4.2 this shows that $\text{cap}_p(B_r, B_R) \simeq \mu(B_R)R^{-p}$, and thus the lower bound in Proposition 4.2 is sharp. Moreover, the above estimate shows that the geometric assumption in Theorem 4.4 and Corollary 4.5 cannot be dropped, that the η -AD property cannot be replaced by global doubling in Theorem 4.1, and also that the assumption $\mu(\{y : d(y, x_0) = R\}) = 0$ cannot be dropped for the limit (5.2) in Proposition 5.1 to hold, even if X supports a global 1-Poincaré inequality.

Example 7.3. Let w be a positive nonincreasing weight function on $X = [0, \infty)$, $d\mu = w dx$ and $x_0 = 0$. Assume that $\mu(B_1) < \infty$. As w is nonincreasing it is easy to see that μ is doubling at x_0 . Let f be an integrable function on X with an

upper gradient g , and let $B = B(x, r) \subset X$ be a ball. Then either $B = (a, b)$ with $0 \leq a < b$ or $B = [a, b)$ with $a = 0 < b$. In either case we have

$$\begin{aligned} \int_B |f - f(a)| d\mu &\leq \frac{1}{\mu(B)} \int_a^b \int_a^t g(x) dx d\mu(t) = \frac{1}{\mu(B)} \int_a^b \int_x^b d\mu(t) g(x) dx \\ &\leq \frac{1}{\mu(B)} \int_a^b r w(x) g(x) dx = r \int_B g d\mu. \end{aligned}$$

It thus follows from Lemma 4.17 in [2] that X supports a global 1-Poincaré inequality.

Moreover, if $0 < \frac{1}{2}R \leq r < R$, then

$$\mu(B_R \setminus B_r) \leq w(r)(R - r) \leq \frac{\mu(B_r)}{r}(R - r) \leq 2\left(1 - \frac{r}{R}\right)\mu(B_R).$$

On the other hand, if $0 < 2r < R$, then

$$\mu(B_R \setminus B_r) \leq \mu(B_R) \leq 2\left(1 - \frac{r}{R}\right)\mu(B_R).$$

Hence μ has the 1-AD property at x_0 .

So far we have just assumed that w is nonincreasing, but now assume that $w(x) = \min\{1, 1/x\}$. If $R > 2$, then by Lemma 3.1,

$$\text{cap}_p(B_{R/2}, B_R) \lesssim \frac{\mu(B_R \setminus B_{R/2})}{R^p} = \frac{\log 2}{R^p},$$

while the right-hand sides (with $r = \frac{1}{2}R$) in Theorem 4.1, Proposition 4.2 and Theorem 4.3 are larger than this when R is large enough, since $\mu(B_R) \rightarrow \infty$, as $R \rightarrow \infty$. In particular it follows that μ cannot be reverse-doubling at x_0 (which also follows directly from $\mu(B_R \setminus B_{R/2}) = \log 2$) and that the reverse-doubling assumption in Theorem 4.1, Proposition 4.2 and Theorem 4.3 cannot be dropped.

Moreover, as $\mu(B_R \setminus B_{R/2}) = \log 2$ and $\mu(B_R) \rightarrow \infty$ as $R \rightarrow \infty$, the inequality in Lemma 4.6 fails in this case, showing that the reverse-doubling assumption in Lemma 4.6 cannot be dropped either.

Write $f(r) = \mu(B_r)$, as in Theorem 6.1. Then $f(r) = 1 + \log r$ for $r \geq 1$. For $\rho > 1$ we have $\rho f'(\rho) = 1 \neq f(\rho)$, so condition (d) in Theorem 6.1 fails. Thus the reverse-doubling assumption in (the last part of) Theorem 6.1 cannot be dropped.

Finally, if we instead let $w(r) = e^{-r}$, then $\mu(B_R) = 1 - e^{-R}$. By Lemma 3.1 we have for $R > 1$ that

$$\text{cap}_p(B_{R/2}, B_R) \leq \text{cap}_p(B_{3R/4}, B_R) \lesssim \frac{\mu(B_R \setminus B_{3R/4})}{R^p} \lesssim \frac{e^{-3R/4}}{R^p}.$$

As $\mu(B_R \setminus B_{R/2}) \simeq e^{-R/2}$ for $R > 1$, this shows that the estimate in Theorem 4.4 fails in this case. Thus the global doubling assumption therein cannot be replaced by assuming that μ is doubling at x_0 . Note that the geometric assumption in Theorem 4.4 is satisfied in this case.

Example 7.4. Let this time w be a positive nondecreasing weight function on $X = [0, \infty)$, $d\mu = w dx$ and $x_0 = 0$. As in Example 7.3 we get that X supports a global 1-Poincaré inequality (estimate using the right end point of the ball instead of the left end point).

Let B be a ball with right end point b . Then $\mu(2B \setminus B) \geq \mu(\{x \in 2B : x > b\}) \geq \frac{1}{2}\mu(B)$, as w is nondecreasing. Hence μ is globally reverse-doubling, i.e. reverse-doubling at every $x \in X$ with uniform constants.

Now let

$$w(x) = \begin{cases} e^{-1/x}/x^2, & 0 \leq x \leq \frac{1}{2}, \\ 4e^{-2}, & x \geq \frac{1}{2}, \end{cases}$$

which is a continuous nondecreasing function such that $\mu(B_R) = e^{-1/R}$ when $0 < R < \frac{1}{2}$. By Lemma 3.1,

$$\text{cap}_p(B_{R/2}, B_R) \leq \text{cap}_p(B_{R/2}, B_{3R/4}) \lesssim \frac{\mu(B_{3R/4})}{R^p}.$$

As

$$\frac{\mu(B_{3R/4})}{\mu(B_R)} = e^{-1/3R} \rightarrow 0, \quad \text{as } R \rightarrow 0+,$$

we see that the lower bound in Proposition 4.2 fails and that the doubling assumption cannot be dropped therein. As $\mu(B_R) \simeq \mu(B_R \setminus B_{R/2})$, this also shows that the estimate in Theorem 4.4 fails in this case. Thus the global doubling assumption therein cannot be replaced by assuming that μ is globally reverse-doubling. Note that the geometric assumption in Theorem 4.4 is satisfied in this case.

In the last example $\mu(B_r) \not\asymp \mu(B_R)$ for $0 < \frac{1}{2}R \leq r < R$ and it is natural to ask if it is possible to get the lower bound $\mu(B_r)r^{-p}$ in Proposition 4.2 without assuming that μ is doubling at x_0 . At least for $p = 1$ this is in fact possible.

Proposition 7.5. *Assume that X supports a 1-Poincaré inequality at x_0 and that μ is reverse-doubling at x_0 with dilation $\tau > 1$. Then*

$$\text{cap}_1(B_r, B_R) \gtrsim \frac{\mu(B_r)}{r}, \quad \text{if } 0 < \frac{R}{2} \leq r < R \leq \frac{\text{diam } X}{2\tau}. \quad (7.2)$$

Example 7.1 shows that the 1-Poincaré assumption cannot be weakened, even if it is assumed globally. Example 7.3 shows that the reverse-doubling assumption cannot be dropped.

Proof. Let u be admissible for $\text{cap}_1(B_r, B_R)$. As in the proof of Theorem 4.1, we get that

$$\begin{aligned} 1 &\lesssim \int_{B_r} |u - u_{B_{\tau R}}| d\mu \leq \frac{\mu(B_{\tau R})}{\mu(B_r)} \int_{B_{\tau R}} |u - u_{B_{\tau R}}| d\mu \\ &\lesssim R \frac{\mu(B_{\tau R})}{\mu(B_r)} \int_{B_{\tau \lambda R}} g_u d\mu \lesssim \frac{r}{\mu(B_r)} \int_{B_R \setminus B_r} g_u d\mu, \end{aligned}$$

and (7.2) follows after taking infimum over all admissible u . \square

References

1. H. AIKAWA and N. SHANMUGALINGAM, Carleson-type estimates for p -harmonic functions and the conformal Martin boundary of John domains in metric measure spaces, *Michigan Math. J.* **53** (2005), 165–188.
2. A. BJÖRN and J. BJÖRN, *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts in Mathematics **17**, European Math. Soc., Zürich, 2011.
3. A. BJÖRN and J. BJÖRN, The variational capacity with respect to nonopen sets in metric spaces, *Potential Anal.* **40** (2014), 57–80.
4. A. BJÖRN, J. BJÖRN, J. GILL and N. SHANMUGALINGAM, Geometric analysis on Cantor sets and trees, to appear in *J. Reine Angew. Math.*, [arXiv:1302.3887](https://arxiv.org/abs/1302.3887).

5. A. BJÖRN, J. BJÖRN and J. LEHRBÄCK, Sharp capacity estimates for annuli in weighted \mathbf{R}^n and in metric spaces, *Preprint*, 2013, [arXiv:1312.1668](#).
6. J. BJÖRN, Poincaré inequalities for powers and products of admissible weights, *Ann. Acad. Sci. Fenn. Math.* **26** (2001), 175–188.
7. S. BUCKLEY, Is the maximal function of a Lipschitz function continuous?, *Ann. Acad. Sci. Fenn. Math.* **24** (1999), 519–528.
8. T. H. COLDING and W. P. MINICOZZI II, Liouville theorems for harmonic sections and applications, *Comm. Pure Appl. Math.* **51** (1998), 113–138.
9. G. DAVID, J.-L. JOURNÉ and S. SEMMES, Opérateurs de Calderón–Zygmund, fonctions para-accrétives et interpolation, *Rev. Mat. Iberoam.* **1**:4 (1985), 1–56.
10. J. GARCÍA-CUERVA and J. L. RUBIO DE FRANCIA, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.
11. P. HAJLASZ, Sobolev spaces on metric-measure spaces, in *Heat Kernels and Analysis on Manifolds, Graphs and Metric Spaces (Paris, 2002)*, Contemp. Math. **338**, pp. 173–218, Amer. Math. Soc., Providence, RI, 2003.
12. J. HEINONEN, *Lectures on Analysis on Metric Spaces*, Springer, New York, 2001.
13. J. HEINONEN, T. KILPELÄINEN and O. MARTIO, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, 2nd ed., Dover, Mineola, NY, 2006.
14. J. HEINONEN and P. KOSKELA, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** (1998), 1–61.
15. J. HEINONEN, P. KOSKELA, N. SHANMUGALINGAM and J. T. TYSON, *Sobolev Spaces on Metric Measure Spaces*, New Math. Monographs **27**, Cambridge Univ. Press, Cambridge, 2015.
16. S. KEITH and X. ZHONG, The Poincaré inequality is an open ended condition, *Ann. of Math.* **167** (2008), 575–599.
17. P. KOSKELA, Removable sets for Sobolev spaces, *Ark. Mat.* **37** (1999), 291–304.
18. P. KOSKELA and P. MACMANUS, Quasiconformal mappings and Sobolev spaces, *Studia Math.* **131** (1998), 1–17.
19. E. ROUTIN, Distribution of points and Hardy type inequalities in spaces of homogeneous type, *J. Fourier Anal. Appl.* **19** (2013), 877–909.
20. N. SHANMUGALINGAM, Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, *Rev. Mat. Iberoam.* **16** (2000), 243–279.
21. N. SHANMUGALINGAM, Harmonic functions on metric spaces, *Illinois J. Math.* **45** (2001), 1021–1050.
22. R. TESSERA, Volume of spheres in doubling metric measured spaces and in groups of polynomial growth, *Bull. Soc. Math. France* **135** (2007), 47–64.